EXPANSION

Taylor's Theorem:- If $f(x+h)$ be a function of variable x (where h is any constant) Then $f(x+h)$ can be extended in ascending power of h such that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$
$$\cdots \frac{h^n}{n!}f^n(h) + \cdots$$

Proof:-

$$\text{Let } f(x+h) = A_0 + A_1h + A_2h^2 + A_3h^3 + \cdots$$
$$\cdots A_nh^n + \cdots$$

OR

Taylor's Theorem:- Let $f(x)$ be a function of x. If the function $f(x+h)$ can be expanded in a convergent series of positive integral power of h, then that expansion is given by

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x)$$
$$\cdots + \frac{h^n}{n!}f^n(x) + \cdots$$

proof: Let us suppose that $f(x)$ possess the derivatives of every order in the closed interval $[a, a+h]$ ~~such and $f(x+h)$~~

Then,

$$f(x+h) = A_0 + A_1h + A_2h^2 + A_3h^3 + \cdots$$
$$\cdots A_nh^n + \cdots \quad -(1)$$

where A_0, A_1, A_2 are the functions of x.

Now,

put $h=0$

Then, on substituting value of $h=0$ we get

$$f(x+h) = A_0 + 0+0+0+0+ \cdots$$

$$f(x) = A_0$$

Now, On diffng eqn (1) w.r.t h we get:

$$f'(x+h) = 0 + A_1 + A_2 2h + A_3 3h^2 + \cdots A_n nh^{n-1} + \cdots \quad \text{--- (2)}$$

put $h=0$ in above eqn we get:

$$f'(x) = A_1 + 0+0+ \cdots +0$$

$$\therefore A_1 = f'(x)$$

Now, On again differentiating equation (2) we get:

$$f''(x+h) = 0 + 2A_2 + A_3 \cdot 3 \cdot 2h + \cdots A_n n(n-1) \cdot h^{n-2} + \cdots \quad \text{--- (3)}$$

put $h=0$ we get

$$f''(x) = 0 + 2A_2 + 0 + 0 + 0 + \cdots$$

$$\therefore A_2 = \frac{f''(x)}{2!}$$

On again diffng eqn (3) we get:

$$f'''(x+h) = 0 + A_3 \cdot 3 \cdot 2 \cdot 1 + A_n n(n-1)(n-2) h^{n-3} + \cdots \quad \text{--- (4)}$$

put $h=0$

$$f'''(x+h) = A_3 \cdot 3 \cdot 2 \cdot 1$$

$$\therefore A_3 = \frac{f'''(x+h)}{6} = \frac{f'''(x)}{3!}$$

Date ______
Page ______

Now, on differentiating equation (4) n times we get.

$$f^{n}(x+h) = 0 + 0 + A_n n(n-1)(n-2)(n-3).h^{n-n}$$
$$= 0 + 0 + A_n n!$$

Now, put $h=0$ we get:

$$f^{n}(x) = A_n n!$$
$$\therefore A_n = \frac{f^{n}(x)}{n!}$$

On substituting the value of $A_0, A_1, A_2, \ldots A_n$ in eqn (1) we get:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \ldots$$
$$\ldots + \frac{f^{n}(x)}{n!}h^{n}.$$

proved

Maclaurin's Theorem: If the function $f(x)$ possess derivatives of every order in $[0, x]$ and can be expressed in a convergent series of positive integral powers of x, then that expansion is given by.

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

proof: Let (1) $f(x)$ be any function of x and can be expanded in convergent series of positive integral of x.

Then, let

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots$$
$$A_n.x^n + \ldots \quad (1)$$

Date ____
Page ____

Now, put $x=0$ in eq^n (1) we get

$$f(0) = A_0 + 0+0+0 \cdots$$

$\therefore A_0 = f(0)$

On differentiating eq^n (1) w.r.to x we get

$$f'(x) = 0 + A_1 + A_2 2x + A_3 \cdot 3 \cdot x^2 + \cdots A_n \cdot x^{n-1} \cdot n \quad \text{(2)}$$

Now, put $x=0$

$$f'(0) = 0 + A_1 + 2 \cdot x \cdot A_2 \times 0 + 0 + 0$$

$\therefore A_1 = f'(0)$

Now, on again differentiating w.r.t x we get

$$f''(x) = 0 + 2A_2 + 3 \cdot 2 \cdot A_3 \cdot x + \cdots A_n n(n-1) \cdot x^{n-2}$$

put $x=0$ we get

$$f''(0) = 2A_2$$

$$\therefore A_2 = \frac{f''(0)}{2!}$$

Similarly $A_3 = \frac{f'''(0)}{3!}$

also $A_n = \frac{f^n(0)}{n!}$

On substituting the value of $A_0, A_1, A_2, A_n \ldots$ in eq^n (1)

$$f(x) = f(0) + xf'(0) + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots \frac{f^n(0) \cdot x^n}{n!}$$ proved

Date ________
Page ________

Failure of Taylor's and Maclaurin series

The expansion of any function $f(x)$ by Taylor's series will not be valid for the following values of x if

i) $f(x)$ or any one of its derivative is infinite.

ii) If the series of the expansion is not convergent.

For example:

$$f(x) = e^{\frac{1}{x}} \text{ where } x \neq 0 \text{ and } f(0) = 0.$$

Solution: $f'(x) = e^{\frac{1}{x}} \cdot (-1) \cdot x^{-2}$

$$= \frac{-1}{x^2} \cdot e^{\frac{1}{x}}$$

At $x = 0$.

$$f'(0) = \infty.$$

$\Rightarrow$ The differential coeffd. $f'(0)$ is ∞ at $x = 0$.

Therefore the expansion of $e^{\frac{1}{x}}$ in ascending order power of x is not possible.

II) $f(x) = \sqrt{x}$

III) $f(x) = \log(x)$

Expansion of some well-known series by means of Maclaurin's Theorem.

i) Expansion of e^x

Let $f(x) = e^x \quad \therefore f(0) = 1$

Now on differentiating successfully.

$$f'(x) = e^x \quad f'(0) = 1$$
$$f''(x) = e^x \quad f''(0) = 1$$
$$f'''(x) = e^x \quad f'''(0) = 1$$
$$f^n(x) = e^x \quad f^n(0) = 1$$

Hence from Maclaurian's theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$$